JOURNAL OF APPROXIMATION THEORY 35, 45–63 (1982)

Lacunary Trigonometric Interpolation on Equidistant Nodes (Convergence)

A. SHARMA

Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

AND

A. K. VARMA

Department of Mathematics, University of Florida, Gainesville, Florida 32611

Communicated by Richard S. Varga

Received January 20, 1981

DEDICATED TO PROFESSOR JOHN TODD ON HIS 70TH BIRTHDAY

1. INTRODUCTION

Recently Riemenschneider *et al.* [2] have studied the problem of the convergence of the trigonometric polynomial $R_n(\theta; f)$, which interpolates a given 2π-periodic function $f(\theta)$ at the nodes $2k\pi/n$ $(k = 0, 1,..., n-1)$ and whose derivatives of orders $m_1, m_2,..., m_q$ are prescribed at these nodes. This is the problem of $(0, m_1,..., m_q)$ interpolation. Earlier Sharma, Smith and Tzimbalario ([4]) (also Cavaretta, Sharma and Varga [1]) had given the necessary and sufficient conditions for $(0, m_1,..., m_q)$ trigonometric interpolation to be uniquely solvable (i.e., regular). The convergence result of [2] is proved under the condition that $f(\theta)$ satisfies the Dini–Lipschitz condition in cases I, III and IV (Theorem A below) and that in case II, $f(\theta)$ satisfies the Zygmund condition. They also remark at the end of their paper that "sometimes the particular cases have better results" and they refer to some of the earlier literature to indicate this. Thus, it was shown about fifteen years back [5] that in the special case of $(0, m_1)$ interpolation with m_1 odd, convergence holds for all 2π-periodic continuous functions. The object of this note is to show that a similar situation prevails in the more general case of $(0, m_1,..., m_q)$ interpolation. Thus we are able to improve Theorem 1 of [2].

0021-9045/82/050045-19$02.00/0

Section 2 will deal with notation and statement of the principal result. In Section 3 we prove some properties of the determinants which we shall need. We devote Section 4 to finding suitable expressions for the fundamental polynomials and in Section 5 we turn to obtain the estimates for sums of the absolute values of the fundamental polynomials. Finally in Section 6 we give the proof of the main theorem.

2. Notation and Main Result

Following earlier practice, we shall say that a trigonometric polynomial $T(\theta) \in \mathscr{E}_M$, if

$$T(\theta) = a_0 + \sum_{\nu=1}^{M} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta), \qquad a_M b_M \neq 0.$$

If, however,

$$T(\theta) = a_0 + \sum_{\nu=1}^{M-1} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) + a_M \cos\left(M\theta + \frac{\varepsilon\pi}{2}\right),$$

with $\varepsilon = 0$ or 1, $a_M \neq 0$, we shall say that $T(\theta) \in \mathscr{E}_{M,\epsilon}$. Let E_q and O_q denote the number of even and odd integers in the set $(m_1, m_2,..., m_q)$. With this notation, the following theorem is known:

Theorem A ([1, 4]). *The problem of* $(0, m_1,..., m_q)$ *trigonometric interpolation on nodes* $2k\pi/n$ $(k = 0, 1,..., n-1)$ *is regular only in the following cases*:

(I) $n = 2m+1$, $q = 2r$, $E_q - O_q = 0$, $T(\theta) \in \mathscr{E}_M$, $M = nr + m$,

(II) $n = 2m+1$, $q = 2r+1$, $E_q - O_q = 1$, $T(\theta) \in \mathscr{E}_{M,0}$, $M = nr + n$,

(III) $n = 2m$, $q = 2r$, $E_q - O_q = 0$, $T(\theta) \in \mathscr{E}_{M,0}$, $M = nr + m$,

(IV) $n = 2m+1$ (or $2m$), $q = 2r+1$, $E_q - O_q = -1$, $T(\theta) \in \mathscr{E}_{M,1}$, $M = nr + n$.

In the sequel we shall be concerned with case IV of the above theorem. Since in this case $E_q - O_q = -1$ and $q = 2r+1$, we shall suppose that $m_1,..., m_r$ are even and $m_{r+1},..., m_q$ are odd. We shall consider the operator $R_n(\theta; f)$ which interpolates $f(\theta)$ and satisfies the conditions:

$$R_n^{(m_\nu)}(\theta_k; f) = \beta_{k\nu}^{(n)} \qquad (\nu = 1, 2,..., q;\ k = 0, 1,..., n-1), \tag{2.1}$$

where $\beta_{k\nu}^{(n)}$ are certain given numbers and $\theta_k = 2k\pi/n$. The fundamental polynomials of this interpolation are given by $\rho_{0,m_j}(\theta)$ which satisfy the conditions:

$$\begin{aligned} \rho_{0,m_j}^{(m_\nu)}(\theta_k) &= 0, \qquad k = 0, 1,..., n-1, \quad \nu \neq j \\ &= 1, \qquad k = 0, \quad \nu = j \\ &= 0, \qquad k \neq 0, \quad \nu = j. \end{aligned} \tag{2.1a}$$

The polynomial $R_n(\theta; f)$ can then be given explicitly by

$$R_n(\theta; f) = \sum_{k=0}^{n-1} f\left(\frac{2k\pi}{n}\right) \rho_{0,0}(\theta - \theta_k) + \sum_{p=1}^{q} \sum_{k=0}^{n-1} \beta_{kp}^{(n)} \rho_{0,m_j}(\theta - \theta_k). \tag{2.2}$$

We shall prove

THEOREM 1. *Suppose* $n = 2m$ *(or* $2m + 1$*),* $q = 2r + 1$, *and* $E_q - O_q = -1$, *with* $m_1, m_2,..., m_r$ *even and* $m_{r+1},..., m_{2r+1}$ *odd and* $M = nr + n$. *If* $f(\theta)$ *is a* 2π*-periodic continuous function, then* $R_n(\theta; f)$ *given by* (2.2) *converges uniformly to* $f(\theta)$ *on the real line, provided the numbers* $\beta_{kj}^{(n)}$ *satisfy the growth conditions*

$$\beta_{kj}^{(n)} = o(n^{m_j}), \qquad j = 1, 2,..., r, \tag{2.3a}$$

$$\beta_{kj}^{(n)} = o(n^{m_j}/\log n), \qquad j = r + 1,..., 2r + 1. \tag{2.3b}$$

The estimates for $\beta_{kj}^{(n)}$ *given by* (2.3a) *and* (2.3b) *cannot be improved.*

Remark. In [2], the authors prove uniform convergence of $R_n(\theta; f)$ to $f(\theta)$ when $f(\theta)$ satisfies the Dini–Lipschitz condition and all the $\beta_{kj}^{(n)}$'s satisfy the conditions

$$\beta_{kj}^{(n)} = o(n^{m_j} \log n), \qquad j = 1, 2,..., 2r + 1.$$

Thus Theorem 1 is an improvement on the result of [2] in two respects:

(i) the Dini–Lipschitz class in [2] has been replaced by the class of 2π-periodic continuous functions, and

(ii) the freedom of $\beta_{kj}^{(n)}$ for $j = 1, 2,..., r$ has been increased as in (2.3a).

The explicit form of the fundamental polynomials $\rho_{0,m_j}(\theta)$ requires the use of the determinant $\Delta_\nu(M, q)$ of order $q + 1$ which is given by

$$\Delta_\nu(M, q) = \begin{vmatrix} 1 & \cdots & 1 & \cdots & 1 \\ (-M+\nu)^{m_1} & \cdots & (-M+\nu+n)^{m_1} & \cdots & (-M+\nu+qn)^{m_1} \\ \vdots & & \vdots & & \vdots \\ (-M+\nu)^{m_q} & \cdots & (-M+\nu+n)^{m_q} & \cdots & (-M+\nu+qn)^{m_q} \end{vmatrix}. \tag{2.4}$$

Let $\Delta_\nu(j, z)$ denote the determinant obtained from $\Delta_\nu(M, q)$ by replacing the row corresponding to m_j (with $m_0 = 0$) by

$$(z^{-M+\nu} \quad z^{-M+\nu+n} \quad \cdots \quad z^{-M+\nu+qn}).$$

Then it is known [2] that the fundamental polynomials $\rho_{0,m_j}(\theta)$ are given by the formula

$$\rho_{0,m_j}(\theta) = \frac{i^{-m_j}}{n} \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(j, z)}{\Delta_\nu(M, q)}, \qquad z = e^{i\theta} \qquad \text{(Case I)}. \tag{2.5}$$

In all other case, we have

$$\rho_{0,m_j}(\theta) = \frac{i^{-m_j}}{n} \left[\frac{1}{2} \frac{\Delta_0(j, z)}{\Delta_0(M, q)} + \sum_{\nu=1}^{n-1} \frac{\Delta_\nu(j, z)}{\Delta_\nu(M, q)} + \frac{1}{2} \frac{\Delta_n(j, z)}{\Delta_n(M, q)} \right]. \tag{2.6}$$

3. Determinants $A_{\nu p}^{(j)}$

Denote the minors of the elements of the $(j+1)$th row of $\Delta_\nu(M, q)$ by $A_{\nu 1}^{(j)}, \ldots, A_{\nu, q+1}^{(j)}$. Then we have

$$\Delta_\nu(M, q) = (-1)^j \sum_{p=1}^{q+1} (-1)^{p-1} A_{\nu p}^{(j)} \tag{3.1}$$

and

$$\Delta_\nu(j, z) = (-1)^j \sum_{p=1}^{q+1} (-1)^{p-1} A_{\nu p}^{(j)} z^{-M+\nu+(p-1)n}. \tag{3.2}$$

In order to find simplified expressions for the fundmamental polynomials and to find improved estimates, we shall need some properties of $A_{\nu p}^{(j)}$. We shall need the known result ([2, Lemma 1]) that the determinant

$$K \begin{pmatrix} t_1, \ldots, t_q \\ m_1, \ldots, m_q \end{pmatrix} = \begin{vmatrix} t_1^{m_1} & t_1^{m_2} & \cdots & t_1^{m_q} \\ \vdots & \vdots & & \vdots \\ t_q^{m_1} & t_q^{m_2} & \cdots & t_q^{m_q} \end{vmatrix} \tag{3.2a}$$

is strictly positive if $0 < t_1 < \cdots < t_q$ and $m_1 < m_2 < \cdots < m_q$ are real numbers.

We shall prove

LEMMA 1. *If* $m_1, \ldots, m_r$ *are even and* $m_{r+1}, \ldots, m_{2r+1}$ *are odd and if*

$\Delta_\nu(M,q)$ and $A_{\nu p}^{(j)}$ are given by (2.4) *and* (3.1), *then the following relations hold true*:

$$\Delta_\nu(M,q)=\Delta_{n-\nu}(M,q), \qquad \nu=0,1,\dots,n-1 \qquad (\text{with } q=2r+1) \quad (3.3)$$

$$A_{\nu,k}^{(j)}=-A_{n-\nu,q+2-k}^{(j)} \qquad (j=0,1,2,\dots,r;\ k=1,2,\dots,q+1), \quad (3.4)$$

$$A_{\nu,k}^{(j)}=A_{n-\nu,q+2-k}^{(j)} \qquad (j=r+1,\dots,2r+1;\ k=1,2,\dots,q+1), \quad (3.5)$$

$$\begin{aligned} A_{0,k}^{(0)}&=0, && k\neq r+2 \\ &=(-1)^{r+1}\Delta_0(M,q), && k=r+2, \end{aligned} \quad (3.6)$$

$$A_{0,r+2-p}^{(j)}=A_{0,r+2+p}^{(j)} \qquad (j=0,1,\dots,r;\ p=0,1,\dots,r), \quad (3.7)$$

$$A_{0,1}^{(j)}=A_{n,q+1}^{(j)}=0, \qquad j=0,1,2,\dots,r. \quad (3.8)$$

Remark. $A_{0,k}^{(j)}\neq 0$ for $2\leqslant k\leqslant q$ and $j=1,2,\dots,r$. Also from (3.4) and (3.6) we have

$$\begin{aligned} A_{n,k}^{(0)}&=0, && k\neq r+1 \\ &=(-1)^{r}\Delta_0(M,q), && k=r+1. \end{aligned} \quad (3.9)$$

Proof. We shall first prove (3.4). A sample row of $A_{\nu k}^{(j)}$ (which is a determinant of order q $(=2r+1)$ with jth row and kth column of $\Delta_\nu(M,q)$ left out) has the elements

$$(-nr-n+\nu)^{m_p}, (-nr+\nu)^{m_p},\dots, (-nr+\nu+(k-2)n)^{m_p}, (-nr-n+\nu+kn)^{m_p},\dots, (nr+\nu)^{m_p}. \quad (3.10)$$

The corresponding row of $A_{n-\nu,q+2-k}^{(j)}$ has the elements

$$(-nr-\nu)^{m_p}, (-nr-\nu+n)^{m_p},\dots, (-nr-\nu+(q-k)n)^{m_p}, (-nr-\nu+(q-k+2)n)^{m_p},\dots, (nr+n-\nu)^{m_p}. \quad (3.11)$$

For $p=0,1,\dots,j-1,j+1,\dots,r$, m_p is even so that row (3.10) is obtained from (3.11) by writing it in the reverse order. However, for $p\geqslant r+1$, since m_p is odd, the row (3.10) is obtained from (3.11) after writing it in the reverse order and then multiplying the row by (-1). Thus $A_{\nu k}^{(j)}$ is obtained from $A_{n-\nu,q+2-k}^{(j)}$ by writing all the columns in the reverse order and multiplying each of the last $r+1$ rows by (-1). Thus

$$\begin{aligned} A_{\nu k}^{(j)}&=(-1)^{r+1}(-1)^{r}A_{n-\nu,q+2-k}^{(j)}, && j\leqslant r \\ &=(-1)^{r}(-1)^{r}A_{n-\nu,q+2-k}^{(j)}, && j\geqslant r+1, \end{aligned}$$

which proves (3.4) and (3.5).

Identity (3.3) follows from (3.1) with $j=0$ on using (3.4).

When $\nu = 0$, the $(r+2)$th column of $\Delta_0(M, q)$ is $(1\ 0 \cdots 0)^T$ from which we get (3.6).

For $j = 0$, (3.8) follows from (3.6). For $j \geqslant 1$, it is enough to prove that $A_{01}^{(j)} = 0$, since $A_{n,q+1}^{(j)} = 0$ follows from (3.4). A sample row if $(-1)^r A_{01}^{(j)}$ is

$$(nr)^{m_p} \cdots n^{m_p} \qquad n^{m_p} \cdots (nr)^{m_p}, \qquad \text{for} \quad 1 \leqslant p \leqslant r, \quad p \neq j$$

and

$$-(nr)^{m_p} \cdots -(n)^{m_p} \qquad n^{m_p} \cdots (nr)^{m_p}, \qquad \text{for} \quad r+1 \leqslant p \leqslant q.$$

By elementary column operations, we have

$$(-1)^r A_{01}^{(j)} = \begin{vmatrix} P & Q \\ R & 0 \end{vmatrix},$$

where 0 is a null-matrix of order $(r+1) \times r$. Equation (3.8) now follows by Laplace expansion in terms of the last $r+1$ columns.

Finally, to prove (3.7), we again see that for $j = 0$, (3.7) follows from (3.6). For $j \geqslant 1$, the identity follows on observing that $A_{0,r+2+p}^{(j)}$ and $A_{0,r+2-p}^{(j)}$ differ from each other only in one column and its location. By elementary row operations, it is easy to see that

$$A_{0,r+2+p}^{(j)} = A_{0,r+2-p}^{(j)} = (-2)^{r-1} B_{j,r-p} \cdot C \cdot n^{-m_j + \Lambda},$$

where

$$C = \begin{bmatrix} 1^{m_{r+1}} & \cdots & (r+1)^{m_{r+1}} \\ \vdots & & \vdots \\ 1^{m_q} & \cdots & (r+1)^{m_q} \end{bmatrix},$$

$\Lambda = \sum_{i=1}^{q} m_i$, and $B_{j,r-p}$ is the minor of the element in the jth row and $(r-p)$th column of the determinant

$$\begin{vmatrix} r^{m_1} & \cdots & 1^{m_1} \\ \vdots & & \vdots \\ r^{m_r} & \cdots & 1^{m_1} \end{vmatrix}.$$

LEMMA 2. *Under the hypotheses of Theorem* 1, *the following estimates hold*:

$$\frac{A_{\nu l}^{(0)}}{\Delta_\nu(M, q)} = O(1) \qquad (\nu = 1, 2, \ldots, n-1;\ l = 1, 2, \ldots, q+1), \tag{3.12}$$

$$\frac{A_{1,r+2+p}^{(0)}}{\Delta_1(M, q)} = \frac{A_{1,r+2-p}^{(0)}}{\Delta_1(M, q)} = O(n^{-1}), \qquad p = 1, \ldots, r, \tag{3.13}$$

$$\delta_\nu^2\left(\frac{A^{(0)}_{\nu,r+2+p}}{\Delta_\nu(M,q)}\right)=\delta_\nu^2\left(\frac{A^{(0)}_{\nu,r+2-p}}{\Delta_\nu(M,q)}\right)=O(n^{-2}),\qquad p=0,1,...,r, \tag{3.14}$$

where $\delta_\nu^2 g(\nu)=g(\nu+1)-2g(\nu)+g(\nu-1)$.

Proof. Since the $(r+2)$th column of $\Delta_\nu(M,q)$ is $(1\ \nu^{m_1}\cdots\nu^{m_q})^T$, all the determinants $A^{(0)}_{\nu l}$ for $l\neq r+2$ will contain the column $(\nu^{m_1}\ \nu^{m_2}\cdots\nu^{m_q})^T$. Thus $A^{(0)}_{\nu,l}=n^\Lambda\cdot G(\alpha)$ with $\alpha=\nu/n$, where $G(\alpha)$ is a polynomial in α of degree $\leqslant\Lambda$ with coefficients which do not depend on $n(\Lambda=\sum_1^q m_k)$.

It is known [2] that $\Delta_\nu(M,q)=n^\Lambda\Phi(\alpha)$, where $\Phi(\alpha)$ is a polynomial in α $(=\nu/n)$ of fixed degree, with coefficients independent of n and with $\Phi(\alpha)\neq 0$ for $0\leqslant\alpha\leqslant 1$. It therefore follows that

$$\frac{A^{(0)}_{\nu,l}}{\Delta_\nu(M,q)}=\frac{G(\alpha)}{\Phi(\alpha)},$$

which proves (3.12). Also

$$\delta_\nu^2\left(\frac{A^{(0)}_{\nu,r+2+p}}{\Delta_\nu(M,q)}\right)\leqslant\frac{c_1}{n^2}\max_{0\leqslant\alpha\leqslant 1}\frac{d^2}{d\alpha^2}\left(\frac{G(\alpha)}{\Phi(\alpha)}\right)\leqslant\frac{c_2}{n^2},$$

where c_1, c_2 are certain constants independent of n. This proves one part of (3.14). The same reasoning applies to the second part of (3.14).

In order to prove (3.13), we note that

$$A^{(0)}_{1,r+2+p}=B\cdot n^\Lambda,$$

where B is a determinant whose $(r+2)$th column consists of the elements $(n^{-m_1}\ n^{-m_2}\cdots n^{-m_r}\ -n^{-m_{r+1}}\cdots -n^{-m_q})^T$. If $\mu=\min_k(m_k)$, then

$$\left|\frac{A^{(0)}_{1,r+2+p}}{\Delta_1(M,q)}\right|\leqslant c_3 n^{-\mu}\leqslant c_3 n^{-1},$$

which proves a part of (3.13). Similarly we get the second part of (3.13).

Remark. Following the same reasoning as above, it can be proved that

$$\frac{A^{(j)}_{\nu l}}{\Delta_\nu(M,q)}=O(n^{-m_j})\qquad(\nu=1,2,...,n-1;\ j=0,...,q;\ l=1,2,...,q+1), \tag{3.12a}$$

$$\frac{A^{(j)}_{1,r+2+p}}{\Delta_1(M,q)}=O(n^{-1-m_j}), \tag{3.13a}$$

$$\delta_\nu^2\left(\frac{A^{(j)}_{\nu,r+2+p}}{\Delta_\nu(M,q)}\right)=\delta_\nu^2\left(\frac{A^{(j)}_{\nu,r+2-p}}{\Delta_\nu(M,q)}\right)=O(n^{-2-m_j})\qquad(p=0,1,...,r;\ j=1,...,q). \tag{3.14a}$$

LEMMA 3. *If* $\Delta_0^j(M,q)=\sum_{p=1}^{q+1} A_{0,p}^{(j)}$, *where* $A_{0,p}^{(j)}$ *are defined by* (3.1), *then for* $j=r+1,\ldots,q$, *we have*

$$\frac{|\Delta_0^j(M,q)|}{\Delta_0(M,q)}=c_1 n^{-m_j}, \qquad c_1>0. \tag{3.15}$$

Proof. From (3.1) it is clear that $\Delta_0^j(M,q)$ is a determinant obtained from $\Delta_0(M,q)$ by replacing the $(j+1)$th row by the following row of $+1$'s and -1's occurring alternatively:

$$(1 \quad -1 \quad 1 \quad \cdots \quad (-1)^{2r+1}) \tag{3.16}$$

$$n^{m_j-\Lambda}\,\Delta_0^j(M,q)=\begin{vmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ (r+1)^{m_1} & \cdots & 1^{m_1} & 0 & 1^{m_1} & \cdots & r^{m_1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (r+1)^{m_r} & \cdots & 1^{m_r} & 0 & 1^{m_r} & \cdots & r^{m_r} \\ -(r+1)^{m_{r+1}} & \cdots & -1^{m_{r+1}} & 0 & 1^{m_{r+1}} & \cdots & r^{m_{r+1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -(r+1)^{m_q} & \cdots & -1^{m_q} & 0 & 1^{m_q} & \cdots & r^{m_q} \end{vmatrix}$$

$j+1$th row

where the $(j+1)$th row is given by (3.16). Performing some elementary column operations and then transposing the first column in front of the last column and moving the $(j+1)$th row into the $(r+2)$th position, we see that

$$n^{m_j}\,|\Delta_0^j(M,q)|=2^{r+1}\,|S|\,n^{\Lambda},$$

where S is a determinant of the form $\left|\begin{smallmatrix} 0 & Q \\ P & R\end{smallmatrix}\right|$, where P is a $r\times r$ matrix, 0 is a null matrix and R is a $r\times(r+1)$ matrix with all elements zero except in the last column which is

$$((r+1)^{m_{r+1}} \quad \cdots \quad (r+1)^{m_{j-1}} \quad (r+1)^{m_{j+1}} \quad \cdots \quad (r+1)^{m_q})^T.$$

More precisely, we have

$$P=\begin{bmatrix} r^{m_{r+1}} & \cdots & 1^{m_{r+1}} \\ \vdots & & \vdots \\ r^{m_q} & \cdots & 1^{m_q} \end{bmatrix}, \qquad Q=\begin{bmatrix} 1^{m_1} & \cdots & (r+1)^{m_1} \\ \vdots & & \vdots \\ 1^{m_r} & \cdots & (r+1)^{m_r} \\ c_1 & \cdots & c_{r+1} \end{bmatrix},$$

where the row $(r^{m_j}\cdots 1^{m_j})$ is missing in P and where $c_\nu=(-1)^{r+\nu-1}\{1+(-1)^{\nu-1}\}/2$ $(\nu=1,2,\ldots,r+1)$.

By Laplace expansion we see that

$$|\Delta_0^j(M, q)| = n^{-m_j+\Lambda} 2^{r+1} |P| |Q|.$$

By (3.2a), $|\text{Det } P| > 0$ and using Laplace expansion and (3.2a), we have $|\text{Det } Q| > 0$. This proves (3.15), since $\Delta_0(M, q) = O(n^\Lambda)$.

4. The Fundamental Polynomials $\rho_{0,m_j}(\theta)$ ($j = 0, 1,..., q$)

We shall obtain simplified expressions for the fundamental polynomials $\rho_{0,m_j}(\theta)$. We shall prove

Lemma 4. *Under the hypothesis of Theorem* 1, *we have the following expressions for* $\rho_{0,0}(\theta)$ *and* $\rho_{0,m_j}(\theta)$:

$$\rho_{0,0}(\theta) = \frac{1}{n}\left[1 + 2\sum_{p=1}^{r+1} (-1)^{r-p+1} \sum_{v=1}^{n-1} \frac{A_{v,r+2-p}^{(0)}}{\Delta_v(M, q)} \cos(np - v)\theta\right]. \quad (4.1)$$

For $j = 1, 2,..., r$, *we have*

$$\rho_{0,m_j}(\theta) = \frac{(-1)^{r+1+j} i^{m_j}}{n}\left[\frac{A_{0,r+2}^{(j)}}{\Delta_0(M, q)} + 2\sum_{p=1}^{r} \frac{(-1)^p A_{0,r+2-p}^{(j)}}{\Delta_0(M, q)} \cos np\theta\right.$$
$$\left. + 2\sum_{p=1}^{r+1} (-1)^p \sum_{v=1}^{n-1} \frac{A_{v,r+2-p}^{(j)}}{\Delta_v(M, q)} \cos(np - v)\theta\right] \quad (4.2)$$

and for $j = r + 1,..., q$, *we have*

$$\rho_{0,m_j}(\theta) = \frac{i^{1+m_j}(-1)^j}{n}\left[\sum_{p=1}^{q+1} (-1)^p \frac{A_{0,p}^{(j)}}{\Delta_0(M, q)} \sin n(r + 2 - p)\theta\right.$$
$$\left. + \sum_{p=1}^{q+1} (-1)^p \sum_{v=1}^{n-1} \frac{A_{vp}^{(j)}}{\Delta_v(M, q)} \sin\{n(r + 2 - p) - v\}\theta\right]. \quad (4.3)$$

Proof. In order to prove (4.1), we begin with (2.6) with $j = 0$ and $z = e^{i\theta}$. From (3.2), (3.3) and (3.4), we see easily that

$$\frac{1}{2}\frac{\Delta_0(0, z)}{\Delta_0(M, q)} + \frac{1}{2}\frac{\Delta_n(0, z)}{\Delta_n(M, q)} = 1. \quad (4.4)$$

Similarly, we see that

$$\sum_{v=1}^{n-1} \frac{\Delta_v(0, z)}{\Delta_v(M, q)} = \frac{1}{2}\sum_{v=1}^{n-1} \frac{\Delta_v(0, z) + \Delta_{n-v}(0, z)}{\Delta_v(M, q)}. \quad (4.5)$$

Since from (3.2), (3.3), and (3.5), we have

$$\frac{1}{2}[\Delta_\nu(0,z)+\Delta_{n-\nu}(0,z)] = \sum_{p=1}^{q+1} (-1)^{p-1} A_{\nu p}^{(0)} \cos\{n(r+2-p)-\nu\}\theta$$
$$= \sum_{p=1}^{r+1} + \sum_{p=r+2}^{2r+2} (-1)^{p-1} A_{\nu p}^{(0)} \cos\{n(r+2-p)-\nu\}\theta$$
$$= 2\sum_{p=1}^{r+1} (-1)^{r-p+1} A_{\nu,r+2-p}^{(0)} \cos(np-\nu)\theta,$$

we see the validity of (4.1) from (4.4) and (4.5).

In order to prove (4.2), we again start with (2.6) for $j = 1, 2,..., r$ and see that using (3.3) and (3.4) we get

$$\frac{1}{2}\frac{\Delta_0(j,z)}{\Delta_0(M,q)} + \frac{1}{2}\frac{\Delta_n(j,z)}{\Delta_n(M,q)}$$
$$= (-1)^j \sum_{p=1}^{q+1} (-1)^{p-1} \frac{A_{0,p}^{(j)}}{\Delta_0(M,q)} \cos\{n(r+2-p)\theta\}$$
$$= (-1)^{r+j+1} \frac{A_{0,r+2}^{(j)}}{\Delta_0(M,q)} + (-1)^j \sum_{\substack{p=1 \\ p\neq r+2}}^{q+1} (-1)^{p-1} \frac{A_{0,p}^{(j)}}{\Delta_0(M,q)} \cos\{n(r+2-p)\theta\}.$$

If we split the last summation into two (one from 1 to $r+1$ and the other from $r+3$ to $2r+2$) and simplify using (3.7), we get

$$\frac{1}{2}\frac{\Delta_0(j,z)}{\Delta_0(M,q)} + \frac{1}{2}\cdot\frac{\Delta_n(j,z)}{\Delta_n(M,q)}$$
$$= (-1)^{r+j+1}\left[\frac{A_{0,r+2}^{(j)}}{\Delta_0(M,q)} + 2\sum_{p=1}^{r} (-1)^p \frac{A_{0,r+2-p}^{(j)}}{\Delta_0(M,q)} \cos np\theta\right]. \quad (4.6)$$

Similarly, from (3.3) we easily get

$$\sum_{\nu=1}^{n-1} \frac{\Delta_\nu(j,z)}{\Delta_\nu(M,q)} = \frac{1}{2}\sum_{\nu=1}^{n-1} \frac{\Delta_\nu(j,z)+\Delta_{n-\nu}(j,z)}{\Delta_\nu(M,q)} = S_1 + S_2, \quad (4.7)$$

where

$$S_1 = (-1)^j \sum_{\nu=1}^{n-1} \frac{1}{\Delta_\nu(M,q)} \sum_{p=1}^{r+1} A_{\nu p}^{(j)} \cos\{n(r+2-p)-\nu\}\theta$$

and

$$S_2 = (-1)^j \sum_{\nu=1}^{n-1} \frac{1}{\Delta_\nu(M,q)} \sum_{p=r+2}^{2r+2} (-1)^{p-1} A_{n-\nu,p}^{(j)} \cos\{n(r+2-p)-n+\nu\}\theta.$$

Since $\{\cos n(r+2-p)-n+\nu\}\theta=\cos\{n(p-r-1)-\nu\}\theta$, we see easily on using (3.7) that

$$S_2=S_1=(-1)^{r+1+j}\sum_{p=1}^{r+1}(-1)^p\sum_{\nu=1}^{n-1}\frac{A^{(j)}_{\nu,r+2-p}}{\Delta_\nu(M,q)}\cos(np-\nu)\theta. \qquad (4.8)$$

The expression (4.2) now follows from (4.6), (4.7) and (4.8). Proof of (4.3) is analogous and is left out.

5. Estimates for $\sum_0^{n-1}|\rho_{0,m_j}(\theta-\theta_k)|$ $(j=0,1,\ldots,r)$

In order to obtain the estimates for these sums, we recall the Fejér kernel and express $\rho_{0,m_j}(\theta-\theta_k)$ in terms of the Fejér kernel. Let $t_{N,k}$ denote the known Fejér kernel (Zygmund [8, Vol. II, p. 21]), where

$$t_{N,k}=1+\frac{2}{N}\sum_{j=1}^{N}(N-j)\cos j(\theta-\theta_k)$$

$$=\frac{1}{N}\left[\sin N(\theta-\theta_k)/2\Big/\sin(\theta-\theta_k)/2\right]^2, \qquad (5.1)$$

where $\theta_k=2k\pi/n$ $(k=0,1,\ldots,n-1)$. It is known that

$$\sum_{k=0}^{n-1}t_{N,k}=n, \qquad t_{1,k}=1 \qquad (5.2)$$

and it is easy to verify that for $N\geqslant 1$, we have

$$2\cos N(\theta-\theta_k)=(N+1)\,t_{N+1,k}-2Nt_{N,k}+(N-1)\,t_{N-1,k}. \qquad (5.3)$$

We now prove

LEMMA 5. *The following representation holds for* $\rho_{0,0}(\theta-\theta_k)$,

$$\rho_{0,0}(\theta-\theta_k)=\sigma_{1,k}+\sigma_{2,k}+\sigma_{3,k},$$

where

$$\sigma_{1,k}=\frac{1}{n}\sum_{p=0}^{r}(-1)^{r-p+1}\sum_{\nu=1}^{n-1}\delta_\nu^2\left(\frac{A^{(0)}_{\nu,r-p+2}}{\Delta_\nu(M,q)}\right)\cdot(np-\nu)\,t_{np-\nu,k},$$

$$\sigma_{2,k}=\frac{1}{n}\sum_{p=0}^{r}(-1)^{r-p+1}\left\{\frac{A^{(0)}_{1,r+2+p}+A^{(0)}_{1,r+2-p}}{\Delta_1(M,q)}\right\}(np)\,t_{np,k}, \qquad (5.5)$$

$$\sigma_{3,k}=\frac{A^{(0)}_{1,1}}{\Delta_1(M,q)}(r+1)\,t_{nr+n,k}$$

and $\delta_\nu^2\,g(\nu)=g(\nu+1)-2g(\nu)+g(\nu-1)$.

Proof. Starting with (4.1) in Lemma 4 with $\theta-\theta_k$ instead of θ and using (5.3), we see that

$$\sigma_{0,0}(\theta-\theta_k)=\frac{1}{n}\left[1+\sum_{p=1}^{r+1}(-1)^{r-p+1}\{S_{1,p}+S_{2,p}+S_{3,p}\}\right],$$

where

$$S_{1,p}=\sum_{v=1}^{n-1}\frac{A^{(0)}_{v-1,r-p+2}}{\Delta_v(M,q)}(np-v+1)\,t_{np-v+1,k},$$

$$S_{2,p}=-2\sum_{v=1}^{n-1}\frac{A^{(0)}_{v,r-p+2}}{\Delta_v(M,q)}(np-v)\,t_{np-v,k}$$

and

$$S_{3,p}=\sum_{v=1}^{n-1}\frac{A^{(0)}_{v,r-p+2}}{\Delta_v(M,q)}(np-v+1)\,t_{np-v+1,k}.$$

It is easily seen that

$$S_{1,p}=\sum_{v=2}^{n}\frac{A^{(0)}_{v-1,r-p+2}}{\Delta_{v-1}(M,q)}(np-v)\,t_{np-v,k}$$

and

$$S_{3,p}=\sum_{v=0}^{n-2}\frac{A^{(0)}_{v+1,r-p+2}}{\Delta_{v+1}(m,q)}(np-v)\,t_{np-v,k}$$

so that

$$\begin{aligned}\rho_{0,0}(\theta-\theta_k)=\sigma_{1,k}+\frac{1}{n}\Bigg[1+\sum_{p=1}^{r+1}(-1)^{r-p+1}\Bigg\{&\frac{A^{(0)}_{n-1,r-p+2}}{\Delta_{n-1}(M,q)}(np-n)\,t_{np-n,k}\\ &+\frac{A^{(0)}_{1,r-p+2}}{\Delta_1(M,q)}(np)\,t_{np,k}-\frac{A^{(0)}_{0,r-p+2}}{\Delta_0(M,q)}(np-1)\,t_{np-1,k}\\ &-\frac{A^{(0)}_{n,r-p+2}}{\Delta_n(M,q)}(np-n+1)\,t_{np-n+1,k}\Bigg\}\Bigg].\end{aligned}$$

From (3.6), $A^{(0)}_{0,r-p+2}=0$ for $p=1,\dots,r+1$ and from (3.9), $A^{(0)}_{n,r-p+2}=0$ for $p=2,\dots,r+1$ and $A^{(0)}_{n,r+1}=(-1)^r\Delta_0(M,q)$, so that we have

$$\sum_{p=1}^{r+1}(-1)^{r-p+1}\frac{A^{(0)}_{0,r-p+2}}{\Delta_0(M,q)}(np-1)\,t_{np-1,k}=0$$

and

$$\sum_{p=1}^{r+1} (-1)^{r-p+1} \frac{A_{n,r-p+2}^{(0)}}{\Delta_n(M,q)} (np-n+1)\, t_{np-n+1,k} = 1.$$

Thus

$$\rho_{0,0}(\theta-\theta_k) = \sigma_{1,k} + \frac{1}{n} \Bigg[\sum_{p=1}^{r+1} (-1)^{r-p+1} \times \left\{ \frac{A_{n-1,r-p+2}^{(0)}}{\Delta_{n-1}(M,q)} (np-n)\, t_{np-n,k} + \frac{A_{1,r-p+2}^{(0)}}{\Delta_1(M,q)} (np)\, t_{np,k} \right\} \Bigg]$$

whence using (3.5), we get (5.4).

LEMMA 6. *The following representation holds for* $\rho_{0,m_j}(\theta-\theta_k)$ *for* $j = 1, 2, \ldots, r$,

$$\rho_{0,m_j}(\theta-\theta_k) = (-1)^{r+1+j}\, i^{m_j} \{\sigma_{1,k}^{(j)} + \sigma_{2,k}^{(j)} + \sigma_{3,k}^{(j)}\}, \tag{5.6}$$

where

$$\begin{aligned} \sigma_{1,k}^{(j)} &= \frac{1}{n} \sum_{p=1}^{r+1} (-1)^p\, \delta_v^2 \left(\frac{A_{v,r+2-p}^{(j)}}{\Delta_v(M,q)} \right) \cdot (np-v)\, t_{np-v,k}, \\ \sigma_{2,k}^{(j)} &= \frac{1}{n} \sum_{p=1}^{r+1} (-1)^p \frac{A_{1,r+2-p}^{(j)}}{\Delta_1(M,q)} (np)\, t_{np,k}, \\ \sigma_{3,k}^{(j)} &= \frac{1}{n} \sum_{p=1}^{r} (-1)^p \frac{A_{1,r+2+p}^{(j)}}{\Delta_1(M,q)} (np)\, t_{np,k}. \end{aligned} \tag{5.7}$$

Proof. Starting with (4.2), we see that

$$(-1)^{r+1+j}\, i^{-m_j} \rho_{0,m_j}(\theta-\theta_k) = S_{1,k}^{(j)} + S_{2,k}^{(j)}, \tag{5.8}$$

where

$$\begin{aligned} S_{1,k}^{(j)} &= \frac{1}{n} \left[\frac{A_{0,r+2}^{(j)}}{\Delta_0(M,q)} + 2 \sum_{p=1}^{r} (-1)^p \frac{A_{0,r+2+p}^{(j)}}{\Delta_0(M,q)} \cos np(\theta-\theta_k) \right], \\ S_{2,k}^{(j)} &= \frac{2}{n} \sum_{p=1}^{r+1} (-1)^p \sum_{v=1}^{n-1} \frac{A_{v,r+2-p}^{(j)}}{\Delta_v(M,q)} \cos(np-v)(\theta-\theta_k). \end{aligned} \tag{5.8a}$$

Using (5.3) we see (after some simple manipulations which we eschew) that

$$S_{2,k}^{(j)} = \sigma_{1,k}^{(j)} + \sum_{l=1}^{4} \beta_{l,k}^{(j)}, \tag{5.9}$$

where

$$\beta_{1,k}^{(j)} = \frac{1}{n} \sum_{p=1}^{r+1} (-1)^p \frac{A_{1,r+2-p}^{(j)}}{\Delta_1(M,q)} (np)\, t_{np,k},$$

$$\beta_{2,k}^{(j)} = \frac{1}{n} \sum_{p=1}^{r+1} (-1)^p \frac{A_{0,r+1+p}^{(j)}}{\Delta_0(M,q)} (np-n+1)\, t_{np-n+1,k},$$

$$\beta_{3,k}^{(j)} = -\frac{1}{n} \sum_{p=1}^{r+1} (-1)^p \frac{A_{1,r+1+p}^{(j)}}{\Delta_1(M,q)} - (np-n)\, t_{np-n,k},$$

$$\beta_{4,k}^{(j)} = -\frac{1}{n} \sum_{p=1}^{n} (-1)^p \frac{A_{0,r+2-p}^{(j)}}{\Delta_0(M,q)} (np-1)\, t_{np-1,k},$$

since $A_{0,1}^{(j)} = 0$ by (3.8).

From (3.8), we have

$$\beta_{2,k}^{(j)} = -\frac{1}{n} \sum_{p=1}^{r} (-1)^p \frac{A_{0,r+2+p}^{(j)}}{\Delta_0(M,q)} (np+1)\, t_{np+1,k} - \frac{1}{n} \frac{A_{0,r+2}^{(j)}}{\Delta_0(M,q)}$$

and

$$\beta_{3,k}^{(j)} = \frac{1}{n} \sum_{p=1}^{r} (-1)^p \frac{A_{1,r+2+p}^{(j)}}{\Delta_1(M,q)} (np)\, t_{np,k}.$$

Since

$$\beta_{1,k}^{(j)} = \sigma_{2,k}^{(j)} + \frac{1}{n} \sum_{p=1}^{r} (-1)^p \frac{A_{0,r+2-p}^{(j)}}{\Delta_0(M,q)} (np)\, t_{np,k}$$

and

$$\beta_{3,k}^{(j)} = \sigma_{3,k}^{(j)} + \frac{1}{n} \sum_{p=1}^{r} (-1)^p \frac{A_{0,r+2+p}^{(j)}}{\Delta_0(M,q)} (np)\, t_{np,k}$$

we see that

$$\begin{aligned}\sum_{l=1}^{4} \beta_{l,k}^{(j)} &= \sigma_{2,k}^{(j)} + \sigma_{3,k}^{(j)} - \frac{1}{n} \frac{A_{0,r+2}^{(j)}}{\Delta_0(M,q)} - \frac{1}{n} \sum_{p=1}^{r} (-1)^p \frac{A_{0,r+2+p}^{(j)}}{\Delta_0(M,q)} \\ &\quad \times \{(np+1)\, t_{np+1,k} - 2np t_{np,k} + (np-1)\, t_{np-1,k}\} \\ &= \sigma_{2,k}^{(j)} + \sigma_{3,k}^{(j)} - S_{1,k}^{(j)} \qquad (5.10)\end{aligned}$$

(on using (5.3) and (5.8a)). From (5.8), (5.9) and (5.10) we get (5.6).

LEMMA 7. *Under the hypotheses of Theorem* 1, *the following estimates hold true*,

$$\sum_{k=0}^{n-1} |\rho_{0,0}(\theta-\theta_k)| \leqslant c_1, \tag{5.11}$$

$$\sum_{|\theta-\theta_k|\geqslant\delta} |\rho_{0,0}(\theta-\theta_k)| = O\left(n^{-1}\operatorname{cosec}^2\frac{\delta}{2}\right), \tag{5.12}$$

$$\begin{aligned}\sum_{k=0}^{n-1} |\sigma_{0,m_j}(\theta-\theta_k)| &\leqslant c_2 n^{-m_j}, \qquad j=1,2,\ldots,r \\ &\leqslant c_3 n^{-m_j}\log n, \qquad j=r+1,\ldots,q.\end{aligned} \tag{5.13}$$

Proof. In order to prove (5.11), we see from (5.4) on using (3.17) in Lemma 2, and (5.3) that

$$\sum_{k=0}^{n-1} |\sigma_{1,k}| = \sum_{p=0}^{r}\sum_{\nu=1}^{n-1} O(n^{-2})(np-\nu) = O(1).$$

Similarly using (3.16) in Lemma 2, we see from (5.4) that

$$\sum_{k=0}^{n-1} |\sigma_{2,k}| = O(1)\sum_{p=0}^{r} p = O(1).$$

In the same manner, we have

$$\sum_{k=0}^{n-1} |\sigma_{3,k}| = O(n^{-1}).$$

This completes the proof of (5.11).

If $|\theta-\theta_k|\geqslant\delta$, then

$$Nt_{N,k} \leqslant c\operatorname{cosec}^2\frac{\delta}{2}$$

so that, as in the case of (5.11), we have

$$\sum_{|\theta-\theta_k|\geqslant\delta} |\rho_{0,0}(\theta-\theta_k)| \leqslant c_4/n\sin^2(\tfrac{1}{2}\delta).$$

The proof of (5.13) for $j=1,2,\ldots,r$ is based on (5.6) and (5.7) and uses the estimates (3.17a). For $j=r+1,\ldots,q$, the inequality has been proved in Riemenschneider *et al.* [2].

Remark. We can show easily that for some ξ, $0 < \xi < 2\pi$

$$\sum_{k=0}^{n-1} |\rho_{0,m_j}(\xi - \theta_k)| > c_3 n^{-m_j}, \qquad j = 0, 1,..., q. \tag{5.14}$$

However, for $j = r + 1,..., q$, when m_j's are odd, we can show that

$$\sum_{k=0}^{n-1} |\rho_{0,m_j}(\pi - \theta_k)| > c_4 n^{-m_j} \log n. \tag{5.15}$$

Thus estimates (5.13) cannot be improved.

In order to prove (5.14), we set

$$g(\theta) = \sum_{k=0}^{n-1} \rho_{0,m_j}(\theta - \theta_k),$$

which is a trigonometric polynomial of order $nr + n$. By Bernstein's inequality for trigonometric polynomials, we have

$$\max_{0 \leqslant \theta \leqslant 2\pi} |g^{(m_j)}(\theta)| \leqslant Cn^{m_j} \max_{0 \leqslant \theta \leqslant 2\pi} |g(\theta)|, \qquad C = (r+1)^{m_j}.$$

If $g(\theta)$ attains its maximum at ξ, we have

$$g(\xi) \geqslant cn^{-m_j} \max_{0 \leqslant \theta \leqslant 2\pi} |g^{(m_j)}(\theta)| \geqslant cn^{-m_j} g^{(m_j)}(\theta_i) = Cn^{-m_j}$$

since $g^{(m_j)}(\theta_i) = 1$ from the properties (2.1a) of the fundamental polynomials. Thus we have shown that

$$\sum_{k=0}^{n-1} |\rho_{0,m_j}(\xi - \theta_k)| \geqslant |g(\xi)| \geqslant cn^{-m_j},$$

which proves (5.14).

The proof of (5.15) is more difficult. We use the formula (4.3) which gives $\rho_{0,m_j}(\theta)$ for $j = r + 1,..., q$. Since $\sin n(r + 2 - p)(\pi - \theta_k) = 0$ and since

$$\sin\{n(r + 2 - p) - v\}(\pi - \theta_k) = -(-1)^{n(r+2-p)} \sin v(\pi - \theta_k),$$

it follows from (4.3) (recall that n is odd) that

$$|\rho_{0,m_j}(\pi - \theta_k)| = \frac{1}{n} \left| \sum_{v=1}^{n-1} \sum_{p=1}^{q+1} \frac{A_{vp}^{(j)}}{\Delta_v(M, q)} \sin v(\pi - \theta_k) \right|$$

so that

$$4 \sin^2 \left(\frac{\pi - \theta_k}{2}\right) |\rho_{0,m_j}(\pi - \theta_k)| = 4 \sin^2 \left(\frac{\pi - \theta_k}{2}\right) \left| \sum_{p=1}^{q+1} S_{n,p}^{(j)} \right| n^{-1},$$

where

$$S_{n,p}^{(j)} = \sum_{\nu=1}^{n-1} \frac{A_{\nu p}^{(j)}}{\Delta_\nu(M,q)} \sin \nu(\pi - \theta_k).$$

Then elementary calculation gives

$$4 \sin^2\left(\frac{\pi-\theta_k}{2}\right) S_{n,p}^{(j)} = -\sum_{\nu=1}^{n-1} \delta_\nu^2 \left(\frac{A_{\nu p}^{(j)}}{\Delta_\nu(M,q)}\right) \sin \nu(\pi-\theta_k)$$
$$+\frac{A_{0,p}^{(j)} + A_{0,q+2-p}^{(j)}}{\Delta_0(M,q)} \sin(\pi-\theta_k).$$

We therefore get

$$\sum_{k=0}^{n-1} |\rho_{0,m_j}(\pi-\theta_k)| = \frac{1}{n} \sum_{k=0}^{n-1} \left| \sum_{p=1}^{q+1} S_{n,p}^{(j)} \right|$$
$$\geqslant \frac{1}{2n} \left| \sum_{p=1}^{q+1} \frac{A_{0,p}^{(j)} + A_{0,q+2-p}^{(j)}}{\Delta_0(M,q)} \right| \sum_{k=0}^{n-1} \left| \cot \frac{\pi-\theta_k}{2} \right|$$
$$-\frac{1}{4n} \sum_{k=0}^{n-1} \sum_{p=1}^{q+1} \sum_{\nu=1}^{n-1} \left| \delta_\nu^2 \left(\frac{A_{\nu p}^{(j)}}{\Delta_\nu(M,q)}\right) \right| \operatorname{cosec}^2\left(\frac{\pi-\theta_k}{2}\right).$$

It is easy to see that

$$\sum_{k=0}^{n-1} \left| \cot \frac{\pi-\theta_k}{2} \right| > cn \log n \qquad \text{and} \qquad \sum_{k=0}^{n-1} \operatorname{cosec}^2 \frac{\pi-\theta_k}{2} = O(n^2).$$

From Lemma 2, we have

$$\left| \sum_{p=1}^{q+1} \frac{A_{0,p}^{(j)} + A_{0,q+2-p}^{(j)}}{\Delta_0(M,q)} \right| = 2 \left| \sum_{p=1}^{q+1} \frac{A_{0,p}^{(j)}}{\Delta_0(M,q)} \right| > cn^{-m_j},$$

and on using (3.17a), we get

$$\sum_{k=0}^{n-1} |\rho_{0,m_j}(\pi-\theta_k)| > cn^{-m_j} \log n - c_1 n^{-m_j} > c_2 n^{-m_j} \log n.$$

This completes the proof of (5.15).

6. Proof of Theorem 1

Since $R_n(\theta; f)$ reproduces constants, we have

$$\sum_{k=0}^{n-1} \rho_{0,0}(\theta-\theta_k) = 1.$$

Therefore from (2.2), we have

$$R_n(\theta;f)-f(\theta)=\sum_{k=0}^{n-1}\{f(\theta_k)-f(\theta)\}\,\rho_{0,0}(\theta-\theta_k)$$
$$+\sum_{j=1}^{q}\sum_{k=0}^{n-1}\beta_{kj}^{(n)}\rho_{0,m_j}(\theta-\theta_k)$$
$$\equiv I_1+I_2.$$

From (5.13) in Lemma 7 and (2.3a), it follows that

$$|I_2|\leqslant\sum_{j=1}^{q}\sum_{k=0}^{n-1}|\beta_{kj}^{(n)}|\,|\rho_{0,m_j}(\theta-\theta_k)|=o(1).$$

From the continuity of $f(\theta)$, we can choose $\delta>0$ such that $|f(\theta)-f(\theta_k)|<\varepsilon$ for $|\theta-\theta_k|\leqslant\delta$, so that

$$|I_1|\leqslant\sum_{|\theta-\theta_k|\geqslant\delta}|f(\theta)-f(\theta_k)|\,|\rho_{0,0}(\theta-\theta_k)|$$
$$+\sum_{|\theta-\theta_k|>\delta}|f(\theta)-f(\theta_k)|\,|\rho_{0,0}(\theta-\theta_k)|,$$

where the first sum is $<c_1\varepsilon$ from (5.11) and the second sum is less that

$$2Kc_2\Big/n\sin^2\frac{\delta}{2},\qquad K\text{ constant},$$

which follows from (5.12) and the boundedness of $f(\theta)$. Thus

$$|R_n(\theta;f)-f(\theta)|\leqslant o(1)+c_1\varepsilon+\frac{2Kc_2}{n\sin^2(\delta/2)},$$

which shows that $R_n(\theta;f)$ converges uniformly to $f(\theta)$ on the real line.

Remark. From (5.14) and (5.15), it follows that estimates (2.3a) and (2.3b) for the number $\beta_{kj}^{(n)}$ cannot be improved from o to O.

Remark. In cases I and III of Theorem A, it was shown earlier [2] that $R_n(\theta;f)$ converges to $f(\theta)$ provided $f\in C_{2\pi}$ and satisfies the Dini–Lipschitz condition. It is possible to prove that the Dini–Lipschitz cannot be relaxed. Further, in case II it was proved earlier [2] that $R_n(\theta;f)$ converges to $f(\theta)$ provided $f\in C_{2\pi}$ and satisfies the Zygmund condition. It is possible to prove that this is also the best possible class. Proof of these facts are quite long, so we will prove them elsewhere.

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